Dynamics of the three helical vortex system and instability

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INTRODUCTION

Many systems develop helical vortices in their wake (propellers, wind turbines, helicopters). Such flows can be assumed, at least locally, to be helically symmetric, i.e invariant through combined axial translation of distance Δz and rotation of angle $\theta = \Delta z/L$ around the same *z*-axis, where $2\pi L$ is a constant called the helix pitch. Analytical [1] and numerical [2] works describing stationary vortices are mostly restricted to inviscid filaments and patches. Here, we present results from a direct numerical simulation (DNS) code with built-in helical symmetry [4]. This code is able to simulate the viscous dynamics of distributed vorticity profiles, it contains in a simple way the effects of 3D vortex curvature and torsion, and allows one to reach higher Reynolds numbers when compared to a full 3D DNS.

In this framework, the long-time (or equivalently far-wake) dynamics of regularly spaced helical vortices is investigated. In this article, we focus on the case of three identical vortices, and simulate their dynamics as their pitch and Reynolds number is varied. This fundamental work is indeed motivated by the case of wind turbine wakes, which are known to be dominated by helical tip and root vortices. At rated wind velocity, the reduced pitch *L* of tip vortices is related [5] to the rotor radius *R* and to the tip-speed ratio λ (which is the ratio between the tangential blade-tip velocity and the wind speed) by $L/R = \sqrt{2}/(3\lambda)$: for typical values $\lambda = 5 - 10$, this yields values as low as 0.05 - 0.1, but smaller as well as far larger values can be reached for other wind speeds. Typical Reynolds numbers $Re = \Gamma/\nu$ based on the circulation Γ of the vortices (ν denotes the kinematic viscosity) are of several million. In these systems, the real flow conditions are made far more complex because of the turbulent atmospheric boundary layer and of coherent structures possibly shed by an upstream turbine, especially in farms. Accurate modelling can be improved if the vortex dynamics and the transition in such complex flows is understood at a fundamental level, and this can be achieved only at the cost of severe simplifications. The present study is done in this spirit. Here, the Reynolds number does not exceed 10^4 , the effects

of nonuniform incoming flow conditions are disregarded and, as a first step, the root vortices are not taken into account. This allows us to focus on basic helical vortex interactions occuring in this system. At large L/R, a "classical" three-vortex merging takes place, which somewhat resembles the twodimensional two-vortex merging. When L/R is reduced, it takes more and more time for the vortices to merge, as their rotation speed around the system axis is slowed down by self-induced vorticity effects. This phenomenon is explained by following the interplay between vorticity and streamfunction in the co-rotating frame of reference [3], and tracking the locus of hyperbolic points of the streamfunction. At low L/R-values, typically less than 1, the exponential instability described by Okulov [6,7] is obtained, resulting in various grouping and merging scenarii at the nonlinear stage of evolution. At intermediate L/R-values of the order of 1, only viscous diffusion acts, resulting in a, slow, viscous type of merging.

Other types of instabilities which are fully threedimensional are not described within this helical framework. The helical code run on a short period of time allows one to generate a quasi-steady flow state which may then be used to investigate such instabilities.

NUMERICAL CODE WITH HELICAL SYMMETRY

Governing equations

A flow displays *helical symmetry* of *helix pitch* $2\pi L$ along a given axis if its velocity field is invariant under the combination of an axial translation of Δz and a rotation of angle $\Delta z/L$ around the same axis. The flow characteristics are identical along the helical lines $\theta - z/L = \text{const.}$ L > 0 corresponds to a right-handed helix and L < 0 to a left-handed helix. A scalar field is helically symmetric if it depends on only two space variables r and $\varphi \equiv \theta - z/L$. Helical symmetry for a vector field \boldsymbol{u} can be expressed as follows:

$$\boldsymbol{u} = u_r(r, \boldsymbol{\varphi}, t)\boldsymbol{e}_r(\boldsymbol{\theta}) + u_{\boldsymbol{\varphi}}(r, \boldsymbol{\varphi}, t)\boldsymbol{e}_{\boldsymbol{\varphi}}(r, \boldsymbol{\theta}) + u_B(r, \boldsymbol{\varphi}, t)\boldsymbol{e}_B(r, \boldsymbol{\theta})$$
(1)

where the orthonormal Beltrami basis (see fig. 1) is defined as

$$\boldsymbol{e}_{B}(r,\theta) = \boldsymbol{\alpha}(r) \left[\boldsymbol{e}_{z} + \frac{r}{L} \boldsymbol{e}_{\theta}(\theta) \right], \quad \boldsymbol{e}_{r}(\theta), \quad \boldsymbol{e}_{\varphi}(r,\theta) = \boldsymbol{e}_{B} \times \boldsymbol{e}_{r}$$
(2)

with

$$\alpha(r) = \left(1 + \frac{r^2}{L^2}\right)^{-\frac{1}{2}}, \quad 0 \le \alpha(r) \le 1.$$
(3)

A general incompressible helical flow can be expressed with only two scalar fields as:

$$\boldsymbol{u} = u_B(r, \boldsymbol{\varphi}, t) \, \boldsymbol{e}_B + \boldsymbol{\alpha}(r) \nabla \boldsymbol{\psi}(r, \boldsymbol{\varphi}, t) \times \boldsymbol{e}_B \tag{4}$$



Figure 1: Local helical basis.

where $\psi(r, \phi, t)$ is a streamfunction. Its vorticity field can be expressed as follows:

$$\boldsymbol{\omega} = \omega_{B}(r,\boldsymbol{\varphi},t) \, \boldsymbol{e}_{B} + \alpha \nabla \left(\frac{u_{B}(r,\boldsymbol{\varphi},t)}{\alpha} \right) \times \boldsymbol{e}_{B} \,. \tag{5}$$

The global field is given by the two scalar fields $\omega_B(r, \varphi, t)$ and $u_B(r, \varphi, t)$: indeed the streamfunction ψ is slaved to both the component of vorticity ω_B and of velocity u_B along the unit vector \boldsymbol{e}_B by

$$\omega_{\scriptscriptstyle B} = -\mathbb{L}\psi + \frac{2\alpha^2}{L}u_{\scriptscriptstyle B} \tag{6}$$

where the linear operator \mathbb{L} is a generalized Laplace operator:

$$\mathbb{L}(\cdot) = \frac{1}{r\alpha} \frac{\partial}{\partial r} \left(r\alpha^2 \frac{\partial}{\partial r} (\cdot) \right) + \frac{1}{r^2 \alpha} \frac{\partial^2}{\partial \varphi^2} (\cdot) \,. \tag{7}$$

The dynamical equations can be thus formulated within a generalization of the standard 2D $\psi-\omega$ method. The equation for u_B reads as

$$\partial_t u_B + NL_u = VT_u \tag{8}$$

where the nonlinear and viscous terms are given by

$$NL_{u} \equiv \boldsymbol{e}_{B} \cdot [\boldsymbol{\omega} \times \boldsymbol{u}],$$

$$VT_{u} \equiv v \left[\mathbb{L}(\frac{u_{B}}{\alpha}) - \frac{2\alpha^{2}}{L} \omega_{B} \right].$$
(9)

The equation for ω_{B} reads

$$\partial_t \omega_B + N L_\omega = V T_\omega \tag{10}$$

where the nonlinear is given by

$$NL_{\boldsymbol{\omega}} \equiv \boldsymbol{e}_{\boldsymbol{B}} \cdot \nabla \times [\boldsymbol{\omega} \times \boldsymbol{u}], \qquad (11)$$

and the viscous term by

$$VT_{\omega} \equiv -v \, \boldsymbol{e}_{\scriptscriptstyle B} \cdot \nabla \times [\nabla \times \boldsymbol{\omega}] = v \left[\mathbb{L}(\frac{\omega_{\scriptscriptstyle B}}{\alpha}) - \left(\frac{2\alpha^2}{L}\right)^2 \omega_{\scriptscriptstyle B} + \frac{2\alpha^2}{L} \mathbb{L}(\frac{u_{\scriptscriptstyle B}}{\alpha}) \right]. \tag{12}$$

The boundary conditions are regularity conditions at the axis and potential flow conditions at the outer circular boundary. As variable $\varphi = \theta - z/L$ is 2π -periodic, the numerical code uses Fourier series along that direction, and second order finite differences in the radial direction. The time advance is performed using second order backward discretisation of the temporal derivative. Nonlinear terms appear explicitly through second order Adams–Bashforth extrapolation whereas the viscous terms are treated implicitly. More details can be found in [4].

Here we simulate the evolution of three identical helical vortices of circulation Γ . Their maximum vorticity is at distance R_0 from the axis, and are equally distributed along the azimuth. Each vortex has an initial small core size a_0 and a pitch $2\pi L$. It would be possible to make quantities dimensionless using the helix radius R_0 as space scale, quantity R_0^2/Γ as time scale. The physical problem would then depend on three dimensionless parameters, namely the Reynolds number $Re = \Gamma/\nu$, and the two ratios L/R_0 and a_0/R_0 . However, the problem can be made generic and dependent only on two parameters if one considers the vortex dynamics starting from a singular helical vortex of radius R_{\star} , pitch L and core size $a_{\star} = 0$ at a certain time origin, say t_{\star} . Selecting different core sizes a_0 as initial conditions for the simulation at t = 0 then amounts to perform a shift of t_{\star} to different points in the past. This procedure, common for rectilinear vortices in 2D vorticity dynamics, is extended here to helical vortices. In the following we thus adopt the quantities R_{\star} and R_{\star}^2/Γ as space and time scales. The dynamics is governed by the two parameters Re and the reduced pitch $\overline{L} \equiv L/R_{\star}$. The initial condition at t = 0 is arbitrarily chosen with core size $\bar{a}_0 \equiv a_0/R_{\star} = 0.2$ and radius $\bar{R}_0 \equiv R_0/R_{\star} = 1$. At the very beginning of the simulation, the radius abruptly increases by less than 1% as the initial condition is no equilibrium state. How the vorticity and velocity distributions are built and the time t_{\star} computed is explained in the following section. Hereafter, all quantities are dimensionless, and, for sake of simplicity, we drop the bars above the corresponding variables.

Generic initial conditions for a set of helical vortices

When defining an initial condition with finite core size, it is thus important to ensure that this state results from the time evolution of an initial singular helical vortex line. First, within the helical symmetry, the conservation of vortex circulation Γ and axial momentum Π_z leads to

$$\int \omega_z \, \mathrm{d}S = \Gamma \,, \quad \int r \omega_\theta \, \mathrm{d}S = \frac{R_\star^2 \Gamma}{L} \,. \tag{13}$$

Let us choose the distribution $\tilde{\omega}_{B}$ in the (r, φ) plane, which corresponds to a Gaussian helical vorticity profile of size a_{0} in a plane orthogonal to the singular filament. When $L \to \infty$ such a solution is an inviscid equilibrium which diffuses via diffusion from a singular filament. When L is finite it is an approximate inviscid equilibrium.

In order to fully determine the flow field, it is necessary to determine the helical velocity distribution u_B . A possible initial condition is such that $u_B/\alpha = \Gamma/(2\pi L)$. This is the case when the vorticity field is everywhere tangent to helical lines. In the inviscid framework, it is known that it remains so. When viscosity is present, this does not hold anymore, and it can be shown that a gradient of u_B/α is generated via viscous coupling between ω_B and u_B . Let us define the function f as

$$f(r,\phi)\equiv \frac{u_{\scriptscriptstyle B}}{lpha}-\frac{\Gamma}{2\pi\,L}$$

It can be established that in the limit of small f, the following relationship holds:

$$f(r,\boldsymbol{\varphi}) = -\frac{2(t-t_{\star})}{L\,Re}\alpha\omega_{\scriptscriptstyle B}(r,\boldsymbol{\varphi})\,.$$

For the generation of the initial condition, we hence assume f to be proportional to $\alpha \omega_B$. We then seek two normalisation constants C and D such that

$$\alpha \omega_{\scriptscriptstyle B} = C \alpha \tilde{\omega}_{\scriptscriptstyle B}$$
 and $f = D \alpha \tilde{\omega}_{\scriptscriptstyle B}$.

These constants are obtained using the conservation laws (13). In order to compute the time origin t_{\star} corresponding to the singular vortex state, we use another conservation law linked to the angular momentum:

$$\int f \, \mathrm{d}S = -\frac{2(t-t_\star)}{L\,Re}\,,$$

that directly yields t_{\star} when applied to the initial condition at t = 0.

The computation is done on dimensionless variables hence $\Gamma = 1$, $R_0 = 1$, $a_0 = 0.2$. The numerical domain is a disk of radius $R_{\text{ext}} = 3$, meshed by $N_r \times N_{\theta}$ grid points. For Reynolds number Re = 5000 and 10000, one choses $N_r = 512$ and $N_{\theta} = 384$. When Re = 1000, these values can be reduced to $N_r = 256$ and $N_{\theta} = 192$.

MERGING OF LARGE PITCH VORTICES

A typical case: L = 2, Re = 5000

In this section, we consider three helical vortices with large pitch, typically $L \ge 2$, and describe the merging process. Fig. 2 displays the helical vorticity ω_B and velocity u_B/α components in the z = 0 plane for several times during the simulation at L = 2 and Re = 5000. Also plotted are the streamlines in the frame rotating with the vortex system, obtained as isocontours of the co-rotating streamfunction ψ_R . These figures can be discussed in association to fig. 3a and 4a which characterize the motion in the z = 0 plane of the point with maximum helical vorticity ω_B , more specifically its radial position $r_{max}(t)$ and its angular velocity $\Omega(t)$. In a first phase, the vortices rotate (see snapshots at $t - t_* = 156$) counterclockwise and grow in size through viscous diffusion. Around a critical time $t_1 - t_* = 363$, the vortices enter a second phase of



Figure 2: Isocontours of (a) ω_B (colored and filled) and ψ_R (white lines) and (b) u_B/α (colored and filled) and ψ_R , at $t - t_* = 156,376,456,956$. Simulation for L = 2 and Re = 5000.

the dynamics, namely a motion towards the center (see snapshots at $t - t_* = 376$), while their angular velocity drastically increases. As there is a continuous shift from phase 1 to phase 2, we use the geometrical construction shown on fig. 3a to define t_1 . As for the case of two-vortex merging, the second phase stops when the vortices are at a certain distance from the center, here for $t - t_* \approx 400$. A third phase ($400 < t - t_* < 600$) then begins with radial oscillations while the vortices keep on expanding (see snapshots at $t - t_* = 456$). This expansion leads to an azimuthal overlap of the vortices and to an eventual axisymmetric corona of helical vorticity. This feature is absent for two-vortex merging where a single central vortex is formed. A fourth phase then begins ($600 < t - t_* < 816$) where the maximum of vorticity inside the corona gently drifts towards the axis, as the asymptotic state is Gaussian (fifth phase). This phenomenon is seen in fig. 3a where a plateau without any oscillation is present as phase 4, which has no counterpart in the case of two-vortex merging.

Influence of the Reynolds number

The influence of the Reynolds number on the dynamics is shown on fig. 3b. Increasing the Reynolds number has several effects:

• Phase 1: the initial diffusion phase 1 is longer as viscous diffusion is diminished, and it is observed from the simulations that the duration of this phase $\Delta t_1 = t_1 - t_*$ is proportional to *Re* at a fixed value of *L*.



Figure 3: Radial position r_{max} of the vorticity maximum as a function of time $t - t_{\star}$ (a) for $L = \infty, 3, 2.5, 2$ and Re = 5000, (b) for L = 2 and Re = 1000, 5000, 10000.

- Phase 2: it is seen that the distance $r_{\max}(t_2)$ at the end of the radial compression phase 2 weakly depends on the Reynolds number, and this phase is shorter for higher Reynolds numbers.
- Phase 3: the frequency of the oscillations during phase 3 do not depend on the Reynolds number, and they are less damped at high *Re*.
- Phase 4: the duration of this purely diffusive phase is directly proportional to the Reynolds number.

Influence of the helical pitch

On fig. 3a, it can be seen that decreasing the pitch L from its infinite 2D value at constant Reynolds number Re has a marked slowdown effect on the merging process. One may be tempted to attribute this slowdown process to the fact that the vortices rotate at a weaker angular velocity as L is decreased, as depicted in fig. 4a. This reduced rotating speed comes from the increasing role of the self-induced velocity which tends to make vortices rotate clockwise. This effect is at the origin of the longer time scales observed as L is decreased, but not in a straightforward fashion, as explained below.



Figure 4: (a) Rotation rate Ω of the position of maximum vorticity with respect to time $t - t_{\star}$ for $L = \infty, 3, 2, 1.5, 1, 0.5$ and Re = 5000. The circle denotes a state for which the maximum vorticity is at this axis, a square denotes a symmetry breaking instability. (b) Squared core size of individual vortices a^2 as a function of time $t - t_{\star}$ for $L = \infty, 3, 2.5, 2, 1.5$ and Re = 5000. The dashed-line shows the 2D diffusion law $a^2(t) = a_0^2 + 4 t/Re$.

It should be first noticed that changing L has no significant effect on the time evolution of the vortex core size during phase 1 (see fig. 4b). The evolution law remains close to the 2D one $a^{2}(t) = a_{0}^{2} + 4 t/Re$. Yet, the critical core size $a(t_{1})$ at which phase 2 begins increases as L is decreased. An explanation can be found by recalling the way the twodimensional vortex merging works in the two-vortex case: there, the convective merging phase 2 begins while a significant amount of vorticity has escaped the closed atmosphere of the two vortices, and begins to form filaments in the surrounding fluid. A similar scenario takes place here, as shown on fig. 5: at critical time t_1 , vorticity has filled the atmosphere of the three vortices, and begins to escape into the peripheral rotating fluid through the hyperbolic points of the corotating streamfunction such as H₁, because of viscous diffusion. The subsequent formation of filaments is believed to be associated to the convective phase 2 whereby vortices are radially pushed towards the axis. It has been shown [8] that, for different values of L and Re, the pertinent parameter for convective merging is not the ratio $a/r_{\rm max}$, but rather the ratio between the core size a and the distance $d \equiv E_1H_1$ between the vortex center and the outer hyperbolic point (see fig. 5). Now, as L is decreased, the rotation speed decreases also, causing the hyperbolic points of the co-rotating streamfunction to move away from the axis, i.e. d increases. It is observed from the simulations that, at critical time, the ratio a/d remains near the value 0.54 ± 0.05 for all L and Re (this value is 0.36 for 2 helical vortices). Hence the critical core size increases together with d as L is decreased, which makes phase 1 last longer.

Fig. 5 also shows the topology of the flow in the vicinity of the axis: contrarily to the twovortex case, a small central region is present, here of triangular shape. This structure seems



Figure 5: Isocontours of ω_B (colored and filled) and ψ_R (white lines) for L = 2 and Re = 10000 near critical time before the convective phase 2. The streamline pattern displays two types of hyperbolic points and two types of elliptic points.

robust — in the two-vortex case, we have here a single hyperbolic point [8] — since it prevents the merging of the vortices at the end of the oscillation phase 3 (see fig. 2 at $t - t_* = 456$).

DIFFUSIVE MERGING AT INTERMEDIATE PITCH

At intermediate pitch $L \approx 1.5$, the rotation speed of the system is weak so that the hyperbolic points such as H₁ are situated far from the axis (typically at distance r > 2), so that neither filamentation, nor convective merging occurs. The behaviour of the system is thus dominated by viscous diffusion and successive vortices smoothly merge (fig. 6a), forming an axisymmetric corona of vorticity at a radial distance from the axis of order unity, since $r_{\text{max}} \approx 1$.

INSTABILITY OF SMALL PITCH VORTICES

At small pitches, an array of helical vortices is known to be unstable [5,6] with respect to displacement modes, a phenomenon responsible for the destabilisation of propeller and wind turbine wakes. Okulov [6] showed that the critical pitch under which such instability occurs for the three vortex case with zero core size is $L_c = 1.132$. In the simulation at L = 1.5 of fig. 6a, the system is a priori stable with respect to Okulov's threshold. In order to check this, the simulation has been launched for the same parameter values, with an initial perturbation on the position of the 3 vortices (one of them has been radially pushed by 0.001, and one other azimuthally displaced by



Figure 6: (a) Isocontours of ω_B and ψ_R at $t - t_* = 260,560,660,960$. Simulation for L = 1.5 and Re = 5000 without initial perturbation. (b) Isocontours of ω_B and ψ_R at $t - t_* = 261,361,461,661$. Simulation for L = 1.5 and Re = 5000, starting from a state in which the vortex positions have been perturbed by an amount 0.001.

0.001). The results are depicted in fig. 6b. It seems that an instability is active, and its effects are felt from $t - t_* = 300$, causing two of the vortices to merge (see snapshot at $t - t_* = 361$). After an exchange of vorticity (see snapshot at $t - t_* = 461$), the system adopts a stable configuration with two thick-cored helices (see snapshot at $t - t_* = 661$). One possibility is that the finite core size significantly alters the instability threshold, but this remains to be ascertained.

At the pitch value L = 1, the system should definitely be unstable. However, if no perturbation is initially set and the growth rate is small enough, the system may diffuse into an axisymmetric helical sheet before the Okulov instability develops significantly. This is illustrated on fig. 7, up to time $t - t_* = 572$. It can be also observed that the newly formed sheet is unstable with respect to an azimuthal perturbation m = 1 (see at $t - t_* = 402$). This leads to a destruction of the sheet (around $t - t_* = 452$) and the system asymptotically converges towards one single helical vortex with a thick core (see at $t - t_* = 852$). However, two reservations can be made: (a) at larger Reynolds numbers, the instability may become active before the sheet forms, (b) the restriction of the study to a helically symmetrical flow with fixed L presumably affects the way the sheet destabilizes: an instability mode is selected, but other waves with different pitches may be more unstable.

If a perturbation of small amplitude is initially set on the system for L = 1, then the instability is rapidly felt on the dynamics of the three vortices. This is illustrated on fig. 8, near time $t - t_{\star} = 122$. One of the helical vortices is then strongly stretched and merges with one of the others (see $t - t_{\star} = 142$ and t = 162.) This is reminiscent of the vortex grouping observed in some experiments [9]. The subsequent evolution is also interesting: the smaller vortex in turn



Figure 7: Isocontours of ω_B and ψ_R at $t - t_* = 172, 372, 402, 422, 452, 872$. Simulation for L = 1 and Re = 5000 without initial perturbation.

gets stretched by the bigger one and they merge together (see $t - t_{\star} = 172$ and t = 192). This event yields one strong helical vortex with a very unsteady behaviour.

SUMMARY

In this paper, we investigate the dynamics of three helical vortices with respect to their helical pitch. At large pitch, vortex merging occurs that bears many analogies with the two-vortex system. The main difference lies in the structure of the axisymmetric state that is reached, namely a corona of vorticity that eventually diffuses smoothly towards a Gaussian vortex. At intermediate pitch $L \approx 1.5$, the system rotates slowly around the axis, and diffusive effects dominate the dynamics. However, a slight shift of the initial vortex positions is able to destabilize the system. At lower values of L, an axisymmetric sheet may also form, but it is found unstable with respect to a m = 1 mode. By contrast, shifting the initial vortex position leads to a rapid destabilisation of the system, and grouping and merging events are observed. This latter case is presumably relevant in real turbine wakes where the nondimensional helical pitch is rather small and large perturbations are due both to incoming flow and rotor geometry.

The three-vortex system can thus reach various asymptotic states either axisymmetric, or helical with one or two vortices depending on the Reynolds number and the pitch. Predicting precise frontiers between the various regimes is not an easy task, because the instability prop-



Figure 8: Isocontours of ω_B and ψ_R at $t - t_* = 82, 122, 142, 162, 172, 192$. Simulation for L = 1 and Re = 5000, starting from a state in which the vortex positions have been perturbed by an amount 0.001.

erties at small pitch strongly depend on the vortex core size. Such sizes are not constant in the viscous regime, and the way they evolve in time is still an open question.

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